Exercise 1.6

CC-MT-8

① ⓐ $\lim_{x\to\infty}\left(\frac{x^3}{3x^2-4}-\frac{x^2}{3x+2}\right)$

$= \lim_{x\to\infty} \frac{x^3(3x+2)-x^2(3x^2-4)}{(3x+2)(3x^2-4)}$

$= \lim_{x\to\infty} \frac{3x^4+2x^3-3x^4-4x^2}{(3x+2)(3x^2-4)}$

$= \lim_{x\to\infty} \frac{2x^3-4x^2}{(3x+2)(3x^2-4)}$

$= \lim_{x\to\infty} \frac{2-4/x}{(3+2/x)(3-4/x^2)} = \frac{2-0}{(3+0)(3-0)} = \boxed{\frac{2}{9}}$

ⓑ $\lim_{x\to 0} \frac{(2x^2+|x|)}{x}$

$\lim_{x\to 0^+} \frac{2x^2+|x|}{x} = 1$; $\lim_{x\to 0^-} \frac{2x^2+|x|}{x} = -1$

~~thus~~ two sided limit are different.

Hence $\lim_{x\to 0} \frac{2x^2+|x|}{x}$ does not exists.

ⓒ $\lim_{x\to 0}\left(\frac{1}{x^2}+3\right) = \boxed{3}$

$\lim_{x\to 0}\left(\frac{1}{x^2}+1\right) = \boxed{1}$

$\lim_{x\to 0}\left\{\left(\frac{1}{x^2}+3\right)-\left(\frac{1}{x^2}+1\right)\right\} = \lim_{x\to 0}\left(\frac{1}{x^2}+3\right) - \lim_{x\to 0}\left(\frac{1}{x^2}+1\right)$

$= 3-1 = \boxed{2}$

Example 13.3.1. *Let* $f(\mathrm{P}) = f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$.

Using ε-δ definition establish: $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2y^2}{x^2+y^2} = 0.$

Solution: Let $\varepsilon > 0$ be given. We are to find a $\delta > 0$ s.t. in some deleted δ-*nbd* of $(0,0)$ for all points (x,y),

$$\left|\frac{x^2y^2}{x^2+y^2} - 0\right| < \varepsilon, \quad \text{or,} \quad \frac{x^2y^2}{x^2+y^2} < \varepsilon.$$

Now clearly,

$$x^2 < x^2+y^2; \quad y^2 < x^2+y^2$$

$$\text{so that} \quad \frac{x^2y^2}{x^2+y^2} < \frac{(x^2+y^2)^2}{x^2+y^2} = x^2+y^2 < \varepsilon$$

$$\text{if} \quad 0 < x^2+y^2 < \delta^2 \text{ where } \delta = \sqrt{\varepsilon}.$$

Thus, we have found out $\delta = \sqrt{\varepsilon}$ to satisfy the requirements of the definition. Therefore,

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{x^2+y^2} = 0.$$

Example 13.3.2. *Establish that* $\lim\limits_{\substack{x\to 0\\ y\to 0}} xy\dfrac{x^2-y^2}{x^2+y^2} = 0.$

Solution: Let $\varepsilon > 0$ be given. To find a δ-*nbd* of $(0,0)(\delta > 0)$ such that in that *nbd* N, $\forall (x,y)$

$$\left|xy\frac{x^2-y^2}{x^2+y^2} - 0\right| < \varepsilon \quad \text{i.e.,} \quad \left|xy\frac{x^2-y^2}{x^2+y^2}\right| < \varepsilon.$$

$$\text{But} \quad \left|xy\frac{x^2-y^2}{x^2+y^2}\right| = |x||y|\frac{|x^2-y^2|}{x^2+y^2}.$$

Clearly, $|x| < \sqrt{x^2+y^2}$, $|y| < \sqrt{x^2+y^2}$ and $\dfrac{|x^2-y^2|}{x^2+y^2} < 1.$

$$\therefore \left|xy\frac{x^2-y^2}{x^2+y^2} - 0\right| = |x||y|\frac{|x^2-y^2|}{x^2+y^2} < \sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2} = x^2+y^2 < \varepsilon$$

$$\text{i.e.,} \quad \left|xy\frac{x^2-y^2}{x^2+y^2} - 0\right| < \varepsilon, \quad \text{if } 0 < x^2+y^2 < \delta^2 \text{ where } \delta = \sqrt{\varepsilon}.$$

Example 13.3.3. *To prove* $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x^2+y^2}$ *does not exist.*

Solution: Let $f(x,y) = \dfrac{xy}{x^2+y^2}$.

The domain of f is the whole xy-plane punctured at the origin $(0,0)$. For existence of limit we are to examine the values of f near $(0,0)$, i.e., in some deleted nbd. of $(0,0)$.

If we introduce polar co-ordinates $x = \rho\cos\theta, y = \rho\sin\theta$, then f takes the form

$$f(x,y) = \frac{xy}{x^2+y^2} = \frac{\rho\cos\theta\cdot\rho\sin\theta}{\rho^2} = \frac{1}{2}\sin 2\theta.$$

Here, of course, $\rho = \sqrt{x^2+y^2}$. As $(x,y)\to(0,0), \rho\to 0$.

From this formula, it is clear that f is constant on each straight line through the origin ($y = x\tan\theta$) and that, in general, the constant is different on different lines. Clearly, then, the values of f cannot be made close to any constant A by simply restricting ρ. Thus, $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy}{x^2+y^2}$ does not exist in this case.

Alternative Method. Put $y = mx$. Then we have

$$f(x,mx) = \frac{m}{1+m^2}.$$

Again, we see that f has different constant values on different lines through the origin; so by letting $(x,y)\to(0,0)$ along a suitable line through the origin, $f(x,y)$ will approach any value k specified at pleasure.

So the values of f cannot be made close to any one constant A in some nbd of $(0,0)$; limit cannot exist in this case. The limit must have to be unique.

Example 13.3.4. *The function f, defined over the whole xy-plane, is given by*

$$f(x,y) = \begin{cases} \dfrac{|x|}{y^2}e^{-|x|/y^2}, & \text{when } y \neq 0; \\ 0, & \text{when } y = 0. \end{cases}$$

Discuss the existence of limit of $f(x,y)$ as $(x,y) \to (0,0)$.

Solution: On the line $x = 0$, we have $f(0,y) = 0$, $\forall y$ and the limit is zero as $(x,y) \to (0,0)$ along this line.

On any other line $x = ay$ through the origin we have

$$f(ay,y) = \frac{|ay|}{y^2}e^{-|ay|/y^2} = \frac{|a|}{|y|}e^{-|a/y|}$$

and the limit of this expression as $y \to 0$ is easily seen to be 0, by using (say) L'Hospital's rule.

Thus as $(x,y) \to (0,0)$ along any straight line through the origin, we have $f(x,y)$ tending to zero.

But see that along the parabola $x = y^2$, we have

$$f(y^2,y) = \frac{y^2}{y^2}e^{-(y^2/y^2)} = e^{-1}.$$

Hence f is always e^{-1} along a parabola passing through the origin and clearly then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ is not unique and the limit, in fact, does not exist.

Example 13.3.5. $f(x,y) = \dfrac{x^2y}{x^4+y^2}$.

Discuss the existence of the limit of $f(x,y)$ as $(x,y) \to (0,0)$.

Solution: Let us consider the double limit as $(x,y) \to (0,0)$. If we approach the origin along any axis, $f(x,y) = 0$ and the limit is zero. If we approach along any line $y = mx$ $(m \neq 0)$ then

$$f(x,y) = \frac{x^2 \cdot mx}{x^4 + m^2x^2} = \frac{mx}{x^2+m^2},$$

which tends to zero as $x \to 0$.

Thus, when we approach the origin along any line $y = mx$, the limit $= 0$. But let us approach the origin along a parabolic path $y = mx^2$; here also $y \to 0$ as $x \to 0$.

$$\text{But,} \quad f(x,y) = f(x, mx^2) = \frac{x^2 \cdot mx^2}{x^4 + m^2x^4} = \frac{m}{1+m^2},$$

which has different values for different m. Hence the double limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} \text{ does not exist.}$$

Example 13.3.6. $\lim(x^2+2y) = 5$, *when* $(x,y) \to (1,2)$. *Verify.*

Solution:

$$\begin{aligned} |x^2+2y-5| &= |x^2-1+2y-4| \leq |x^2-1| + 2|y-2| = |x+1||x-1| + 2|y-2| \\ &< 2\delta + 2\delta \text{ when } 0 < |x-1| < \delta \text{ and } 0 < |y-2| < \delta \text{ if } 0 < \delta \leq 1 \end{aligned}$$

$\therefore$ $|x^2+2y-5| < \varepsilon$, where ε is any given positive number,

if $4\delta < \varepsilon$, i.e., if $\delta < \varepsilon/4$ or 1, whichever is smaller, i.e., $\delta < \min\{\frac{\varepsilon}{4}, 1\}$.

Thus, the definition requirements are met and the statement is TRUE.