1) (a) Is the intersection of infinite no. of opensets open in a topological space? Jusify your answer? Also give an example of a topological space where any intersection of opensets is open.

⇒ No, the intersection of infinite no. of opensets is not open

Let, in $(\mathbb{R}, \tau)$ $\left(1-\frac{1}{n}, 1+\frac{1}{n}\right)$ is an openset but

$\bigcap_{n\geq 1}^{\infty}\left(1-\frac{1}{n}, 1+\frac{1}{n}\right) = \{1\}$ which is not in $(\mathbb{R}, \tau)$.

In a discrete topology, any intersection of opensets is open

(b) Give an example of a mapping from a topological space to another which closed and continuous but not open

⇒ Let $\mathbb{R}$ be taken with usual topolgy and $f: \mathbb{R} \to \mathbb{R}$ be taken as

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

(c) Show that the real line $\mathbb{R}$ with lower limit topology is not second countable.

⇒ See, solution of Dec 2015 / June 2016.

(d) Give an example of a topological space that is T_1 but not T_2 and explain with reasons.

⇒ Let X be an infinite set and let the collection $\mathcal{G}$ of subsets of X be as $\mathcal{G} = \{G \subset X : (X \setminus G) \text{ is a finite set}\}$.

Then $\mathcal{G}$ is a topology on X which is T_1 but not T_2.

Take two members $u, v \in X$ with $u \neq v$. Then $U = X \setminus \{u\}$

respectively such that v is outside U and u is not in V. Hence $(X, \mathcal{G})$ is T_1. If possible, let any two distinct pts x, y in X have T_2 separation. So we find two open sets H and K in X such that $x \in H$, $x \in K$ and $H \cap K = \phi$. clearly $(X \setminus H)$ and $(X \setminus K)$ are finite subsets of X, so X is a finite set because $H \cap K = \phi$. This is a contradiction. Hence the result.

(e) P.T Continuous image of a compact set is compact

=> Let (X, τ) be a compact space and (Y, V) be a topological space, and $f : X \to Y$ be a continuous function. We show that $f(X)$ is compact ... Take $\mathcal{G} = \{G_\alpha\}_{\alpha \in \Delta}$ be an open cover for $f(X)$. So, $f(X) \subset \cup\{G_\alpha : G_\alpha \in \mathcal{G}\}$

$\therefore X \subset f^{-1}\left(\bigcup_{\alpha \in \Delta} G_\alpha\right) = \bigcup_{\alpha \in \Delta} f^{-1}(G_\alpha)$ — (1)

By continuity of f each $f^{-1}(G_\alpha)$ is open in X, and (1) shows that the family $\{f^{-1}(G_\alpha)\}_{\alpha \in \Delta}$ is an open cover for (X, τ) by compactness of which it follows that there are a finite no. of members, say $f^{-1}(G_1)$, $f^{-1}(G_2), \dots f^{-1}(G_n)$ such that $X \subset f^{-1}(G_1) \cup \dots \cup f^{-1}(G_n) = f^{-1}\left(\bigcup_{i=1}^{n} G_i\right)$

So, $f(X) \subset \left(\bigcup_{i=1}^{n} G_i\right)$. So given open cover G for $f(X)$ has a finite sub-cover for $f(X)$.

$\therefore f(X)$ is compact.

(g) When a topological space is called connected? Give an example to show that a subspace of a connected topological space need not be connected.

=> (X, τ) is said to be connected if X is not a union of two non-empty disjoint open sets in X.

• The subset $\mathbb{Q}$ of all rationals in real no. space with usual topology is disconnected.

(f) See solution of Dec 2015 / June 2016 of 1(d).

(a) In a topological space, (X, Υ), for any two sets A, B, prove that $(\overline{A \cup B}) = \bar{A} \cup \bar{B}$.

$\Rightarrow$ Here $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

$\Rightarrow \bar{A} \subseteq \overline{A \cup B}$ and $\bar{B} \subseteq \overline{A \cup B}$ $\quad (\because A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B})$

$\Rightarrow \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ — (i)

Now, we shall show that $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ — (ii)

Then if possible, $x \in \overline{A \cup B}$ and $x \notin \bar{A} \cup \bar{B}$.

Then $x \in \bar{A} \cap \bar{B}$

$\Rightarrow x$ is either an adherent pt of A nor that of B.

$\Rightarrow$ there exist open spheres $S(x, r_1)$ and $S(x, r_2)$ containing no points of A and B respectively.

Let $r = \min\{r_1, r_2\}$. Then $S(x, r)$ contains no pt of A as well as that of B and so open sphere $S(x, r)$ contains no pt of $A \cup B$ and hence x cannot be an adherent pt of $A \cup B$ i.e. $x \in \overline{A \cup B}$, Thus we arrive at a contradiction. Hence

$$x \in \overline{A \cup B} \Rightarrow x \in \bar{A} \cup \bar{B} \quad \text{— (iii)}$$

from (ii) & (iii) we have $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(b)

[redacted] a) Define a filter. P.T a function $f:(X,\tau)\to(Y,\tau')$ is continuous at $x=c\in X$ iff for every filter $\mathcal{F}$ in X which converges to c, $f(\mathcal{F})$ converges to $f(c)$.

⇒ ~~[illegible], Satisfying of~~

⇒ A filter in a topological space (X,τ) is a family $\mathcal{F}$ of subsets of X satisfying (i) If $A\in\mathcal{F}$ and $A\subset B$, then $B\in\mathcal{F}$.

(ii) If $A_1, A_2\in\mathcal{F}$, then $(A_1\cap A_2)\in\mathcal{F}$

(iii) $\phi\notin\mathcal{F}$.

□ The condition is necessary:- Let f be continuous at $x=c\in X$, and let $\mathcal{F}$ be a filter in X converging to c. Then the nbd. system $\mathcal{N}_c$ at $x=c$ is a sub-family of $\mathcal{F}$ i.e. $\mathcal{N}_c\subset\mathcal{F}$. We show that image filter $f(\mathcal{F})$ in Y converges to $f(c)$; that is to show the nbd. system $\mathcal{N}_{f(c)}$ at $f(c)$ as a sub family of $f(\mathcal{F})$ or $\mathcal{N}_{f(c)}\subset f(\mathcal{F})$ — (i)

Let $N_{f(c)}$ be a nbd. of $f(c)$ in Y. i.e $N_{f(c)}\in\mathcal{F}_{f(c)}$. By continuity of f at $x=c$, we find that $f^{-1}(N_{f(c)})$ is a nbd. of c in X and hence by assumption $f^{-1}(N_{f(c)})\in\mathcal{F}$; that means $N_{f(c)}\in f(\mathcal{F})$ and (i) is verified.

The condition is sufficient:- Suppose the condition holds. and f is not continuous at $x=c$. we drive a contradiction.

we find a nbd. $N_{f(c)}$ of $f(c)$ in Y such that no nbd. N_c of c in X satisfies $N_{f(c)} \subset f^{-1}(N_{f(c)})$

$\therefore$ image filter $f(\mathcal{N}_c)$ ~~does~~ does not converge to $f(c)$ in Y, though nbd. filter $\mathcal{N}_c$ converge to c in X. a contradiction. The proof is now complete.

(b) Define a net. In a topological space $(X, \mathcal{J})$ prove that $\{S_n : n \in D\}$ is a net converging to $u \in X$, iff there is a filter in X which converges to u.

$\Rightarrow$ See, Solution of Dec 2013 / June 2014 of 3 (a).

4(a) Show that a topological space is T_1 iff every singleton is closed.

$\Rightarrow$ Let (X, τ) be T_1 and $x \in X$. If $y \in X$ and $y \neq x$ by T_1-separation we find an open set U such that $x \in U$ and $y \notin U$.

Clearly, then y is not a limit pt of $\{x\}$ i.e. $y \notin \{x\}'$ (derived set of $\{x\}$). Of course, $x \notin \{x\}'$. That means no member of X is a ~~no.~~ member of $\{x\}'$. Hence $\{x\}' = \phi$. So $\{\bar{x}\} = \{x\} \cup \{x\}' = \{x\}$

i.e. $\{\bar{x}\}$ is closed.

Conversely, suppose every singleton in (X, τ) is closed. Take $x, y \in X$ with $x \neq y$. So singleton $\{x\}$ is closed, and hence $X \setminus \{x\}$ is an open set containing y (without containing x); and similarly $X \setminus \{y\}$ is an open set containing x (without containing y). Thus (X, τ) is T_1