CC - PH - 07 Unit-1

Problem 1 - Explain the Periodic function and State Fourier's theorem:

Ans:- Any ~~a~~ function which repeats it self regularly over a given function is called periodic function.

Here function $f(x)$ is periodic. i.e. $f(x+p) = f(x)$.

$f(x)$ is a periodic function over an interval p.

The fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Here

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx.$$

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx.$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\, dx.$$

a_0, a_n & b_n are called Fourier co-efficients.

The periodic function $f(x)$ can be represented by the sum of harmonic terms as

$$f(x) = \tfrac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx.$$

2) Describe the complex from of Fourier's theorem.

Ans:- The fourier series for the function $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \cdots (1)$$

Substituting $a_n = c_n \cos \theta_n$ and $b_n = c_n \sin \theta_n$ in eqn (1) we get,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (c_n \cos \theta_n \cos nx + c_n \sin \theta_n \sin nx)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(nx - \theta_n) \quad \cdots (2)$$

where

$$c_n = \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad \theta_n = \tan^{-1} \frac{b_n}{a_n}$$

Similarly, substituting $\cos nx = \frac{e^{inx} + e^{-inx}}{2}$

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

in eqn (1) we get,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} - e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} - e^{-inx}}{2} - i b_n \frac{e^{inx} - e^{-inx}}{2} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n - i b_n}{2} e^{inx} + \frac{a_n + i b_n}{2} e^{-inx} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - i b_n}{2} e^{inx} + \sum_{n=-1}^{-\infty} \frac{a_n + i b_n}{2} e^{inx} \quad \cdots (3)$$

Now put $a_0/2 = c_0$.

$$\left.\begin{aligned} c_n &= 1/2 (a_n - i b_n) \\ c_{-n} &= 1/2 (a_n - i b_n) \end{aligned}\right\} \cdots (4)$$

we get, $f(x) = c_0 + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=-1}^{-\infty} c_{-n} e^{inx}$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \cdots (5)$$

eqⁿ (5) represents the complex form of fourier series and the co-efficient C_n is known as complex fourier co-efficient.

(3) What is odd and even function? Explain with example.

Ans:- A function $f(x)$ is said to be even if $f(-x) = f(x)$

Ex:- x^2, $\cos x$, $\sec x$, $|x|$

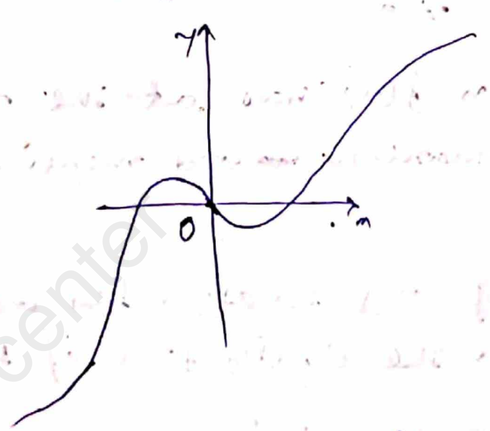
Graphically an even function is symmetrical about the y-axis.

A function $f(x)$ is said to be odd if

$f(-x) = -f(x)$

Ex:- x, $\sin x$, x^3

Graphically an odd function is symmetrical about the origin.



We shall be using the following property of definite integrals is,

$$\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx. \quad \text{when } f(x) \text{ is an even function.}$$

$$= 0 \quad \text{when } f(x) \text{ is odd function.}$$

(4)(a) What are the Dirichlet's conditions in Fourier's series?

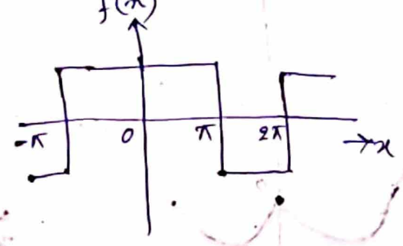
(b) What is the limitations of Fourier's theorem.

Ans:-

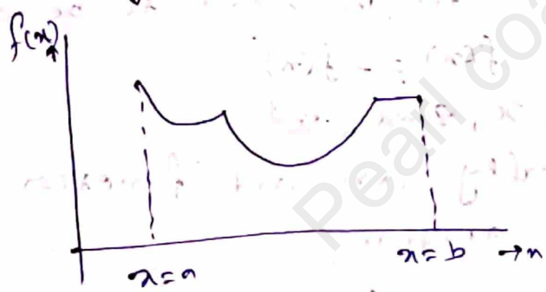
Any function $f(x)$ can be developed as a fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$. where a_0, a_n and b_n are constant.

The conditions are,

(1) The function $f(x)$ is periodic, single valued and finite.



(2) The function $f(x)$ has a finite number of discontinuities in any one period.



(3) The function $f(x)$ has at the most a finite number of maxima ~~ma~~ and minima.

(b) The problem expressing any function $f(x)$ as a Fourier series depends upon the evaluation of the integrals

$$\frac{1}{\pi}\int f(x)\cos nx\, dx \; ; \; \frac{1}{\pi}\int f(x)\sin nx\, dx.$$

Within the limits $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha+2\pi)$ accordingly as $f(x)$ is defined for every value of x in $(0, 2\pi)$, $(-\pi, \pi)$ or $(\alpha, \alpha+2\pi)$

5) $F(x)$ have a Fourier series Expension.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{then}$$

Prove that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

Ans:- The fourier series for $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1).}$$

multiplying both side of (1) by $f(x)$ and integrating from $-\pi$ to π

we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0}{2}\int_{-\pi}^{\pi} f(x) dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} f(x) \cos nx \, dx + b_n \int_{-\pi}^{\pi} f(x) \sin nx \, dx\right] \quad \text{--- (2).}$$

we know from fourier theorem,

$$\int_{-\pi}^{\pi} f(x) dx = a_0.$$

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = a_n.$$

$$\int_{-\pi}^{\pi} f(x) \sin nx \, dx = b_n.$$

from eqn (2) we get,

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)}$$

The above relation is called Parseval relation.